

Queen Mary & Westfield College

UNIVERSITY OF LONDON

MAS204 CALCULUS III

5th May 1998 10.00-12.00

TIME 2 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators may be used in this examination but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

You are reminded of the following, which you may use without proof.

In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the divergence of a vector field $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1$, $h_2 = r$, $h_3 = 1$.

SECTION A Each question carries **8 marks**. You should attempt ALL questions.

A1. In Cartesian coordinates (x, y, z) , evaluate at the point $(1, 2, 3)$

$$(a) \nabla f, \quad (b) \mathbf{V} \cdot \mathbf{F}, \quad (c) \nabla \times \mathbf{F}, \quad (d) \nabla^2 f,$$

where $f = x^2z + y^3$ and $\mathbf{F} = yx^3\mathbf{i} + zy^2\mathbf{j} + xz\mathbf{k}$.

A2. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 6y\mathbf{i} + 7z^2\mathbf{j} + 2x\mathbf{k}$ and C is a path going from $(0, 0, 0)$ to $(2, 4, 8)$ and is described in parametric form as

$$C: \mathbf{r} = t\mathbf{i} + 2t\mathbf{j} + t^3\mathbf{k}$$

A3. Prove that (a) $\mathbf{V} \cdot (\mathbf{a} \times \mathbf{r}) = 0$ and (b) $\mathbf{V} \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$, where $\mathbf{r}$ is the position vector and $\mathbf{a}$ is a constant vector.

A4. For each of the following vector fields $\mathbf{F}$, either find the general solution ϕ satisfying $\mathbf{F} = \nabla\phi$, or show that no such ϕ exists,

$$(a) \mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}, \quad (b) \mathbf{F} = z^2\mathbf{i} + x^2\mathbf{j} + y^2\mathbf{k}.$$

A5. Show that the following are solutions of Laplace's equation $\nabla^2 f = \mathbf{V} \cdot \nabla f = 0$:

$$(a) f = r^2 \cos 2\theta, \quad (b) f = r \sin \theta \sin \phi,$$

where (a) is in cylindrical polar coordinates and (b) is in spherical polar coordinates.

A6. Sketch a few representative streamlines for the following fluid velocity fields:

$$(a) \mathbf{u} = -y\mathbf{i} + x\mathbf{j}, \quad (b) \mathbf{u} = (-x/y)\mathbf{i} + \mathbf{j}.$$

For each of the flows (a) and (b) state whether it is a possible fluid velocity for irrotational flow of an incompressible fluid.

A7. An infinitely long straight wire of constant radius a lies along the z -axis of a cylindrical polar coordinate system and carries a uniformly distributed, steady current of magnitude I . By evaluating the integral $\int_C \mathbf{B} \cdot d\mathbf{r}$ around a suitable closed path C , show that the θ component of the associated magnetic field $\mathbf{B}$ is

$$B_\theta = \frac{\mu_0 I}{2\pi r},$$

where r is the distance from the wire. (You may wish to employ Stokes' Theorem and the time-independent Ampere's Law, $\mathbf{V} \times \mathbf{B} = \mu_0 \mathbf{j}$, where $\mathbf{j}$ is the current density).

SECTION B Each question carries **22 marks**. You may attempt all questions but **only** marks for the best **2** questions will be **counted**.

B1. Consider Laplace's equation $\nabla^2 V = 0$ in the case where V is a function of cartesian coordinates x and y only. Find all possible separable solutions of the form $V(x, y) = X(x)Y(y)$. Find the function $V(x, y)$ satisfying $\nabla^2 V = 0$ in the region $0 < x < 1$, $0 < y < 2$, subject to the boundary conditions $V = \sin(\pi x)$ along $y = 2$ and $V = 0$ on the other three sides.

B2. State the Divergence Theorem.

A vector field $\mathbf{F}$ is given in spherical polar coordinates by

$$\mathbf{F} = r^2 \sin \theta \mathbf{e}_r + 2r^2 \cos \theta \mathbf{e}_\theta - r^2 \sin 2\phi \mathbf{e}_\phi.$$

Show that $\nabla \cdot \mathbf{F} = 4r \sin^2 \phi / \sin \theta$.

Verify that the Divergence Theorem is satisfied by $\mathbf{F}$ over a sphere of radius a centred on the origin.

A second vector field $\mathbf{F}_2$ differs from $\mathbf{F}$ only in its component along $\mathbf{e}_\phi$, and is given by

$$\mathbf{F}_2 = r^2 \sin \theta \mathbf{e}_r + 2r^2 \cos \theta \mathbf{e}_\theta + 2r^2(\sin \theta - 1) \phi \mathbf{e}_\phi.$$

Show that this field DOES NOT satisfy the Divergence Theorem and state the reason for this.

B3. Analysis of Laplace's equation in spherical polar coordinates requires solutions to Legendre's Equation

$$(1 - x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0.$$

Using the method of Frobenius, or otherwise, show that this equation has a polynomial solutions if $\lambda = l(l+1)$, where l is a positive integer. Determine the solution for $l = 3$.

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B4. The vector fields $\mathbf{A}$ and $\mathbf{B}$ satisfy the equation

$$\mathbf{B} = \nabla \times \mathbf{A} .$$

Prove that $\nabla \cdot \mathbf{B} = 0$.

A new vector field $\mathbf{A}'$ is defined by $\mathbf{A}' = \mathbf{A} + \nabla \phi$, where ϕ is a given (but arbitrary) scalar field. Prove that $\mathbf{B}$ is also given by $\nabla \times \mathbf{A}'$.

A vector field $\mathbf{F}$ satisfies

$$\nabla \times \mathbf{F} = f \mathbf{F}$$

where $f(\mathbf{r})$ is a scalar function of position only. Prove that $(\nabla \times \mathbf{F}) \times \mathbf{F} = 0$.
Prove also that if $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F}$ and f must satisfy $\mathbf{F} \cdot \nabla f = 0$.

Show by direct evaluation that for the field

$$\mathbf{B} = B_0 ((\cos az^2)\mathbf{i} + (\sin az^2)\mathbf{j}) ,$$

where B_0 and a are constants, $\nabla \cdot \mathbf{B} = 0$ and $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$.

Find a scalar function f so that

$$\nabla \times \mathbf{B} = f \mathbf{B}$$

and show that $\mathbf{B} \cdot \nabla f = 0$.

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